When the arc-colored line digraph of a Cayley colored digraph is again a Cayley colored digraph

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Abstract. Let $D_\Delta(G)$ be the Cayley colored digraph of a finite group G generated by Δ. The arc-colored line digraph of a Cayley colored digraph is obtained by appropriately coloring the arcs of its line digraph. In this paper it is shown that the group of automorphisms of $D_\Delta(G)$ that act as permutations on the color classes is isomorphic to the semidirect product of G and a particular subgroup of $AutG$. Necessary and sufficient conditions for the arc-colored line digraph of a Cayley colored digraph also to be a Cayley colored digraph are then derived.

1. Definitions and Terminology

For the sake of completeness we define in this section the main concepts used in the rest of the paper.

A *digraph* $D = (V, A)$ consists of a set $V = V(D)$ of points called *vertices* and a set $A = A(D)$ of pairs of different vertices called *arcs*. Thus, neither loops nor parallel arcs are allowed. The *order* of D is its number of vertices $|V|$ and the *size* of D is its number of arcs $|A|$. If (x, y) is an arc from x to y we say that x is *adjacent to* y and also that y is adjacent *from* x. A digraph is *d-regular* when each vertex is adjacent to and from exactly d others. A digraph is *strongly connected* when for any $x, y \in V$ there is a directed path from x to y.

Let $C = \{c_1, c_2, ..., c_d\}$ be a set of elements called *colors*. An *arc-coloring* of a d-regular digraph D is a mapping from $A(D)$ to C such that the d arcs from any vertex x are mapped to different colors, and likewise for the d arcs to x.

An *automorphism* of a digraph D is a permutation of V that preserves adjacency. For an arc-colored digraph, the concept of color preserving automorphism is more appropriate. A *color preserving automorphism* ϕ of an arc-colored digraph must also preserve the arc colors, that is, if (u, v) is an arc colored c_i, the arc $(\phi(u), \phi(v))$ must also be colored c_i. Alternatively, if for $v \in V$ and $c \in C$ we represent by $v \odot c$ the vertex adjacent from v by an arc colored c, we can say that ϕ must satisfy

$$\phi(v \odot c) = \phi(v) \odot c. \tag{1}$$

Both the sets of automorphisms of a digraph and of color preserving automorphisms of an arc-colored digraph are groups. We are mainly concerned here with several automorphism groups of Cayley colored digraphs considered as arc-colored digraphs.

In the line digraph LD of a digraph $D = (V, A)$ each vertex represents an arc of D, that is, $V(LD) = \{uv : (u, v) \in A(D)\}$, and two vertices uv, wz, are adjacent whenever $v = w$. As the order of LD is the size of D, if D is d-regular then the order of LD is d times the order of D.

We finally cite that, given two groups G and H together with a homomorphism $\Pi : H \to AutG$, $\Pi(x) = \pi_x$, the *(external) semidirect product* $G \times H$ is the group with set of elements $\{(g, x) : g \in G, x \in H\}$ and composition rule

$$(g_1, x_1)(g_2, x_2) = (g_1\pi_{x_1}(g_2), x_1x_2),$$

for instance, see [10]. When H is a subgroup of $AutG$ it will be assumed that Π is the canonical imbedding.

2. The Color Permuting Automorphism Group

Let G be a finite group with identity element e and generated by $\Delta = \{a_1, a_2, ..., a_d\} \subseteq G - e$. The *Cayley colored digraph* of G with respect to Δ, denoted $D_\Delta(G)$, is the d-regular digraph whose vertices represent the elements of G, and where there is an arc (g, h) labeled (or colored) a iff $h = ga$ for some $a \in \Delta$. For Cayley colored digraphs the meaning of (1) is that a permutation ϕ of $V(D_\Delta(G))$ is a color preserving automorphism iff $\phi(ga) = \phi(g)a$ for all $g \in G$ and $a \in \Delta$.

If $\mathcal{A}(D_\Delta(G))$ denotes the color preserving automorphism group of $D_\Delta(G)$, a well-known and useful result states that

$$\mathcal{A}(D_\Delta(G)) \cong G \tag{2}$$

for any (that is, independently of the generating set Δ) Cayley colored digraph $D_\Delta(G)$ of a finite group G. Frucht [9] used this result to prove that every finite group is the automorphism group of some graph. For a proof of both results see, for instance, White [11].

A slightly more general concept is due to Chvátal and Sichler [2] who defined a *chromatic automorphism* of a graph D as an automorphism of D that acts as a permutation on the set of colors for every proper minimal coloring of its vertices. Analogously, we define a *color permuting automorphism* of a Cayley colored digraph $D_\Delta(G)$ as an automorphism of $D_\Delta(G)$ that acts as a permutation on the color classes. Hence, a permutation ϕ of $V(D_\Delta(G))$ is a color permuting automorphism of $D_\Delta(G)$ iff there exists a permutation σ of Δ such that

$$(3) \qquad \phi(ga) = \phi(g)\sigma(a)$$

for all $g \in G$ and $a \in \Delta$. Obviously, ϕ is a color permuting automorphism of $D_\Delta(G)$ only when σ is the identity. On the other hand, arc transitive Cayley colored digraphs, that is, digraphs such that for any $g \in G$ and $a, a' \in \Delta$ there is an automorphism ϕ which satisfies $\phi(ga) = \phi(g)a'$, are characterized by Babai in [1].

Let $\mathcal{A}^*(D_\Delta(G))$ denote the group of color permuting automorphisms of $D_\Delta(G)$. Its elements are characterized by the following result.

LEMMA 2.1. *Let ϕ be a permutation of $V(D_\Delta(G))$ and let $H = \{\pi : \pi \in AutG, \pi(\Delta) = \Delta\}$. Then*

$$(4) \qquad \phi \in \mathcal{A}^*(D_\Delta(G)) \Leftrightarrow \phi(h) = g\pi(h), \quad \forall h \in G,$$

for some $g \in G$ and some $\pi \in H$.

PROOF: Any ϕ defined as in (4) is a permutation of $V(D_\Delta(G))$. Moreover, for all $h \in G$ and $a \in \Delta$,

$$\phi(ha) = g\pi(ha) = g\pi(h)\pi(a) = \phi(h)\pi(a),$$

so that (3) is satisfied with $\sigma = \pi|_\Delta$ and hence $\phi \in \mathcal{A}^*(D_\Delta(G))$.

Conversely, let $\phi \in \mathcal{A}^*(D_\Delta(G))$ and call $g = \phi(e)$. Any element $h \in G$ can be written in terms of the generators as $h = x_1x_2...x_m$, $x_i \in \Delta$. Thus using (3) m times we obtain

$$(5) \qquad \phi(h) = \phi(ex_1x_2...x_m) = g\sigma(x_1)\sigma(x_2)\ldots\sigma(x_m)$$

for some permutation σ of Δ. Moreover, σ induces in an obvious way a permutation π of G:

$$(6) \qquad \pi(h) = \sigma(x_1)\sigma(x_2)\ldots\sigma(x_m), \quad \forall h \in G$$

since, by (5),

$$x_1x_2\ldots x_m = y_1y_2\ldots y_n \quad \Leftrightarrow \quad \pi(x_1x_2\ldots x_m) = \pi(y_1y_2\ldots y_n).$$

Of course, π is an automorphism of G that fixes the set Δ, and from (5) and (6) it follows that $\phi(h) = g\pi(h)$ for all $h \in G$.

THEOREM 2.2. *Let $D_\Delta(G)$ be the Cayley colored digraph of the finite group G with respect to Δ, and let H be the subgroup of $AutG$ fixing Δ. Then*

$$\mathcal{A}^*(D_\Delta(G)) \cong G \times H \tag{7}$$

PROOF: Consider the mapping $\Psi : G \times H \to \mathcal{A}^*(D_\Delta(G))$ defined by $\Psi[(g,\pi)](h) = \phi_{g\pi}(h) = g\pi(h)$. It is a homomorphism since

$$\begin{aligned}\Psi[(g_1,\pi_1)(g_2,\pi_2)](h) &= \Psi[(g_1\pi_1(g_2),\pi_1\pi_2)](h) = g_1\pi_1(g_2)\pi_1\pi_2(h)\\ &= g_1\pi_1(g_2\pi_2(h)) = \phi_{g_1\pi_1}\phi_{g_2\pi_2}(h).\end{aligned}$$

Moreover, Ψ is clearly one-to-one and, by Lemma 2.1, onto.

Note that when H is the trivial group $\mathcal{A}^*(D_\Delta(G)) \cong \mathcal{A}(D_\Delta(G))$, so that (7) can be seen as a generalization of (2). Also, since G is (isomorphic to) a normal subgroup of $G \times H$ it follows that $\mathcal{A}(D_\Delta(G))$ is a normal subgroup of $\mathcal{A}^*(D_\Delta(G))$.

3. THE ARC-COLORED LINE DIGRAPH OF A CAYLEY COLORED DIGRAPH

As before, we consider a finite group G generated by $\Delta = \{a_1, a_2, \ldots, a_d\}$. On the other hand, let $H = \{\pi_z : z \in \Delta\}$ be now a set of permutations of Δ such that π_{a_1} is the identity and for all $x, y \in \Delta$ there exists only one z such that $\pi_z(x) = y$, that is, H acts on Δ like a regular group, even though it need not be a group under composition. Without loss of generality we may assume that the elements of Δ are labelled so that $\pi_{a_i}(a_1) = a_i$.

For a given H we define the *arc-colored line digraph* L_HD of the Cayley colored digraph $D = D_\Delta(G)$ as the digraph with set of vertices $\{(g,x) : g \in G, x \in \Delta\}$ and where (g,x) is adjacent to (g',x') by an arc with the color denoted by $\alpha(a_i)$ iff

$$(g,x) \odot \alpha(a_i) = (gx, \pi_x(a_i)) = (g', x'). \tag{8}$$

Each pair (g,x) may be thought of as an arc of D.

Note that, since π_x is a permutation of Δ, the vertices obtained in (8) for all $a_i \in \Delta$ are different and correspond to the d arcs adjacent from (g,x) in D. Analogously, each of the d vertices (ga_j^{-1}, a_j) for $a_j \in \Delta$ is adjacent to (g,x) by an arc colored $\alpha(a_i)$ with $a_i = \pi_{a_j}^{-1}(x)$ since

$$(ga_j^{-1}, a_j) \odot \alpha(a_i) = (ga_j^{-1}a_j, \pi_{a_j}\pi_{a_j}^{-1}(x)) = (g,x),$$

and different arcs have different colors because $\{\pi_{a_j}^{-1}(x) : 1 \leq j \leq d\} = \Delta$. Therefore $L_H D$ is the line digraph of D with a proper arc-coloring.

We next want to explore the structure of $A(L_H D)$. From H it is possible to define a binary operation on Δ:

$$x \oplus y = \pi_x(y) \tag{9}$$

that has the cancellation property. The subset Δ^* of elements $z \in \Delta$ such that $z \oplus (x \oplus y) = (z \oplus x) \oplus y$, for all $x, y \in \Delta$, has a_1 as identity element and is easily seen to be closed under $\oplus$. Therefore, Δ^* with this binary operation is a group. Moreover, there exists $H^* \subseteq H$ such that H^* with the composition of permutations is a group isomorphic to Δ^*, since the bijection

$$\Psi : \Delta^* \to H^*, \Psi(z) = \pi_z \tag{10}$$

satisfies

$$\begin{aligned}\Psi(z_1 \oplus z_2)(x) &= (z_1 \oplus z_2) \oplus x = z_1 \oplus (z_2 \oplus x)\\ &= \pi_{z_1}(\pi_{z_2}(x)) = \Psi(z_1)\Psi(z_2)(x).\end{aligned}$$

The following result, analogous to Lemma 2.1, shows that every element of $A(L_H D)$ corresponds to a color permuting automorphism of $D_\Delta(G)$. As in the preceding section, this allows the identification of H^* with a subgroup of $AutG$, that is, using (6) every $\pi_z \in H^*$ defines an automorphism of G, that we still call π_z. We already use this identification in the statement of the lemma.

Lemma 3.1. *Let Φ be a permutation of $V(L_H D)$. Then $\Phi \in A(L_H D)$ if and only if for all (h, x), $h \in G$, $x \in \Delta$,*

$$\Phi[(h, x)] = (g\pi_z(h), \pi_z(x)) = (g\pi_z(h), z \oplus x)$$

for some $g \in G$ and some $\pi_z \in H^*$.

Proof: For the sufficiency we have

$$\begin{aligned}\Phi[(h, x) \odot \alpha(a_i)] = \Phi[(hx, x \oplus a_i)] &= (g\pi_z(hx), z \oplus (x \oplus a_i))\\ &= (g\pi_z(h)\pi_z(x), (z \oplus x) \oplus a_i)\\ &= (g\pi_z(h), \pi_z(x)) \odot \alpha(a_i) = \Phi[(h, x)] \odot \alpha(a_i).\end{aligned}$$

For the necessity, suppose $\Phi \in \mathcal{A}(L_H D)$. From the construction of $L_H D$ it is clear that Φ corresponds to a permutation of $A(D)$ that preserves arc adjacency (but not necessarily the coloring). Therefore, it induces in the obvious way a permutation ϕ on $V(D)$. It also induces, for every fixed $h \in G$, a bijection π_h^+ $[\pi_h^-]$ from the set of arcs adjacent from [to] $h \in V(D)$, $(h, \Delta) = \{(h, x) : x \in \Delta\}$ $[(h\Delta^{-1}, \Delta) = \{(hx^{-1}, x) : x \in \Delta\}]$ into the set of arcs adjacent from [to] $h' = \phi(h) \in V(D)$, (h', Δ) $[(h'\Delta^{-1}, \Delta)]$. Hence, π_h^+ $[\pi_h^-]$ acts as a permutation of Δ, that for simplicity we still call π_h^+ $[\pi_h^-]$, and Φ can be written in either of the forms

$$\Phi[(h, x)] = (\phi(h), \pi_h^+(x)),$$
$$\Phi[(hx^{-1}, x)] = (\phi(hx^{-1}), \pi_h^-(x)), \tag{12}$$

for all $h \in V(D)$ and $x \in \Delta$.

Since Φ preserves the arc-coloring of $L_H D$,

$$\Phi[(hx^{-1}, x) \odot \alpha(a_i)] = \Phi[(hx^{-1}, x)] \odot \alpha(a_i) \tag{13}$$

for any $h \in G$ and $x, a_i \in \Delta$. With (12) this results in

$$\phi(h) = \phi(hx^{-1})\pi_h^-(x), \tag{14}$$

and

$$\pi_h^+(x \oplus a_i) = \pi_h^-(x) \oplus a_i. \tag{15}$$

Now, for $a_i = a_1$ (15) gives $\pi_h^+(x) = \pi_h^-(x)$ for all x, that is, $\pi_h^+ = \pi_h^-$ and we call both π_h. Then $\pi_h(x \oplus a_i) = \pi_h(x) \oplus a_i$ for all a_i, so that the permutation π_h is completely specified by the image of any element $x \in \Delta$. This, jointly with the fact that $\pi_{hx}^-(x) = \pi_h^+(x)$, implies $\pi_{hx} = \pi_h$ and then $\pi_h = \pi$ for all $h \in G$ since D is strongly connected. Moreover, with $z = \pi(a_1)$ we have

$$\pi(y) = \pi(a_1 \oplus y) = \pi(a_1) \oplus y = z \oplus y = \pi_z(y),$$

hence $\pi = \pi_z$ and $\pi_z \in H^*$ since, for all $x, y \in \Delta$,

$$z \oplus (x \oplus y) = \pi_z(x \oplus y) = \pi_z(x) \oplus y = (z \oplus x) \oplus y.$$

Finally, replacing hx^{-1} by h in (14) we obtain

$$\phi(hx) = \phi(h)\pi_z(x) \tag{16}$$

so that, by virtue of (3), $\phi \in \mathcal{A}^*(D_\Delta(G))$. Then, from (12) and Lemma 2.1 we obtain

$$\Phi[(h,x)] = (\phi(h), \pi_z(x)) = (g\pi_z(h), \pi_z(x))$$

for some $g \in G$.

From Lemma 3.1 we obtain the following result, which is analogous to Theorem 2.2 and is proved similarly.

THEOREM 3.2. *Let $D = D_\Delta(G)$, H and $L_H D$ be as above. Then*

$$\mathcal{A}(L_H D) \cong G \times H^* \tag{17}$$

where $H^* \subseteq H$ is a subgroup of $AutG$.

A consequence of this theorem are necessary and sufficient conditions for the arc-colored line digraph of a Cayley colored digraph also to be a Cayley colored digraph. The sufficiency was studied in [5]. The interest in the arc-colored line digraph of a Cayley colored digraph stems from the following two points.

(a) The permutation networks which are easiest to control are those modelled by Cayley colored digraphs, see for instance [6].

(b) The line digraph technique has proven very useful to generate large digraphs with small diameter and reduced average distance [7,8]. Therefore, the line digraph of a Cayley colored digraph will usually have good order-diameter ratio and, when a Cayley colored digraph, the corresponding permutation network will also have an easy control.

THEOREM 3.3. *The arc-colored line digraph $L_H D$ of the Cayley colored digraph $D = D_\Delta(G)$ of a finite group G is a Cayley colored digraph of a group Ω if and only if H is a subgroup of $AutG$ and then $\Omega \cong G \times H$.*

PROOF: If $L_H D$ is a Cayley colored digraph there exists a group Ω with $|\Omega| = |V(L_H D)| = |H||G|$ elements such that $\mathcal{A}(L_H D) \cong \Omega$. Hence, by Theorem 3.2, $\Omega \cong G \times H$ with H a subgroup of $AutG$.

Conversely, let H be a subgroup of $AutG$. Then the set Δ with the composition rule $\oplus$ defined in (9) is a group isomorphic to H and we can consider the group $G \times \Delta \cong G \times H$ whose elements are in correspondence with the vertices of $L_H D$. By Theorem 3.2,

$$\mathcal{A}(L_H D) \cong G \times \Delta \cong \mathcal{A}(D_\Lambda(G \times \Delta))$$

with $D_\Lambda(G \times \Delta)$ the Cayley colored digraph of $G \times \Delta$ associated to any set Λ of generators. In our case, each color $\alpha(a_i)$ of $L_H D$ can be associated with the element (a_1, a_i) of $G \times H$ since then

$$(g, x) \odot \alpha(a_i) = (g, x)(a_1, a_i) = (g\pi_x(a_1), x \oplus a_i) = (gx, x \oplus a_i)$$

in agreement with (8). Therefore, with $\Lambda = \{(a_1, a_i) : a_i \in \Delta\}$, we have the isomorphism of arc-colored digraphs

$$L_H D \cong D_\Lambda(G \times \Delta).$$

We conclude by considering the particular case of groups generated by two elements. Let $\Delta = \{a, b\}$. Then $H = \{\pi_a, \pi_b\}$, where π_a is the identity and π_b interchanges a and b. For Theorem 3.3 to hold, π_b must be an automorphism of the group G, and this requires in turn the symmetry of the two generators and their relations.

For instance, this is the case for the alternating group A_n with the presentation of Coxeter [3,4]:

$$a^{2s-1} = b^{2s-1} = (ba)^s = e, \quad (b^i a b^{-1} a^{-i})^2 = e \quad (1 \leq i \leq s-1)$$

when $n = 2s$ is even and

$$a^{2s+1} = b^{2s+1} = (ba)^s = e, \quad (b^j a^{-j})^2 = e \quad (2 \leq j \leq s),$$

when $n = 2s + 1$ is odd. For example we can take $a = (1\ 2\ 3 \ldots n-1)$, $b = (2\ 3\ 4 \ldots n)$ when n is even and $a = (1\ 2\ 3 \ldots n)$, $b = (1\ n\ 2\ 3 \ldots n-1)$ when n is odd. In both cases $\pi_b(g) = \sigma g \sigma$ where σ is the involution $\sigma = (1\ n)$. Then $L_H D$ turns out to be the Cayley colored digraph of a group $\Omega \cong A_n \times C_2 \cong S_n$, the symmetric group. The last isomorphism $\psi : A_n \times C_2 \to S_n$ is given by

$$\psi[(g, a)] = g \quad \text{and} \quad \psi[(g, b)] = g\sigma$$

In particular, the generators of S_n are

$$\psi[(a, a)] = a, \quad \text{and} \quad \psi[(a, b)] = a\sigma$$

which gives $(1\ 2 \ldots n-1)$ and $(1\ 2 \ldots n)$ when n is even, and $(1\ 2 \ldots n)$ and $(1\ 2 \ldots n-1)$ when n is odd, respectively and with products from left to right. We can express this result by informally saying that the line digraph of the Cayley colored digraph of the alternating group A_n is the Cayley colored digraph of the symmetric group S_n.

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